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ANALYSIS ON TRANSVERSE IMPACT RESPONSE OF AN UNRESTRAINED TIMOSHENKO BEAM *

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Abstract: *A moving rigid-body and an unrestrained Timoshenko beam, which is subjected to the transverse impact of the rigid-body, are treated as a contact-impact system. The generalized Fourier-series method was used to derive the characteristic equation and the characteristic function of the system. The analytical solutions of the impact responses for the system were presented. The responses can be divided into two parts: elastic responses and rigid responses. The momentum sum of elastic responses of the contact-impact system is demonstrated to be zero, which makes the rigid responses of the system easy to evaluate according to the principle of momentum conservation.*

Key words: unrestrained; Timoshenko beam; tansverse impact; elastic response; rigid response; momentum

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Introduction

As a classical problem, the study on elastic impact problem of structures dates as far back as the early years of the 20th century. Recently, the research on dynamics and control of space structures has revived interest in the propagation of waves in a beam[1]. Large and flexible lattice-type space structures, which are situated in the airless and zero gravity environment, are often constructed by lightweight bars. Because of its length-width ratio being large, the structure can be approximated by a beam. So the study on transient responses (i.e., the impact responses) of a

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beam to impact loadings is very useful for dynamic control of the space structures.

In this paper, the Timoshenko beam is analyzed for the reason that the effects of shear deformation are not neglectful factors for the impact problems of structures. Especially for high-order modes, the effects are very evident. The vibration and wave motion in the Timoshko beam have been studied by many researchers: Boley and Chao investigated solutions for four types of loadings applied to a semi-infinite beam by using the method of Laplace transformation[2]; Miklowitz also used the Laplace transform to obtain the transient responses for infinite length beams and finite length beams[3]. Anderson and Calif gave the general series solution for the flexural vibrations of a finite length beam[4]. Huang discussed the normal solutions for six common types of beams by using the method of variables separation[5]. XING Yu-feng presented the semi-analytical solutions of transverse elastic impact and contact between a mass point and a beam with finite length[6].

A space structure moving freely in the orbit without any supports, if its length-width ratio is large, can be approximately seen as an unrestrained Timoshenko beam. The differences between impact problems of unrestrained and restrained structures lie in that there are rigid responses for the former, e.g., a transverse impact on an unrestrained Timoshenko beam by a moving rigid body may cause rigid responses and elastic responses of the beam. In this paper, the analytical solutions of an unrestrained Timoshenko beam which is subjected to a transverse impact of a moving rigid body are derived by using the generalized Fourier-series method. This method separates the rigid responses of the beam from its total responses, and the momentum sum of elastic responses in the system is demonstrated to be zero, which makes the evaluation of rigid response for the system very simple.

1 Basic Equations and General Solutions

The motion equations for a Timoshenko beam are

$$EI\frac{\partial^2\psi}{\partial x^2} + kAG\left(\frac{\partial y}{\partial x} - \psi\right) - \rho I\frac{\partial^2\psi}{\partial t^2} = 0, \tag{1}$$

$$\rho A\frac{\partial^2 y}{\partial t^2} - kAG\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) = 0. \tag{2}$$

Where $y(x,t)$ is the transverse displacement of the beam; $\psi(x,t)$ is the bending rotation angle of the beam's cross-section; E, G, I, A and ρ are the elastic modulus, shear modulus, inertia moment of the cross-section, cross-sectional area and mass density of the beam, respectively; k is the shape factor of the cross-section.

After rearranging, Eqs.(1) and (2) can be written as

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} - \left(\rho I + \frac{\rho EI}{kG}\right)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 y}{\partial t^4} = 0, \tag{3}$$

$$EI\frac{\partial^4 \psi}{\partial x^4} + \rho A\frac{\partial^2 \psi}{\partial t^2} - \left(\rho I + \frac{\rho EI}{kG}\right)\frac{\partial^4 \psi}{\partial x^2\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 \psi}{\partial t^4} = 0. \tag{4}$$

Let

$$y(\xi,t) = Y(\xi,p)\sin pt, \tag{5}$$

$$\psi(\xi,t) = \Psi(\xi,p)\sin pt, \tag{6}$$

$$\xi = x/L. \tag{7}$$

In which, ξ is dimensionless coordinate along the axis of the beam; p is the angular frequency; L is the length of the beam.

Substituting Eqs. (5) ~ (7) into Eqs. (1) ~ (2), and eliminating the term $\sin pt$, the following expressions can be obtained:

$$s^2 \frac{\partial^2 \Psi}{\partial \xi^2} - (1 - b^2 r^2 s^2)\Psi + \frac{1}{L}\frac{\partial Y}{\partial \xi} = 0, \tag{8}$$

$$\frac{\partial^2 Y}{\partial \xi^2} + b^2 s^2 Y - L\frac{\partial \Psi}{\partial \xi} = 0, \tag{9}$$

where

$$b^2 = \rho A L^4 p^2/(EI), \tag{10}$$

$$r^2 = I/AL^2, \tag{11}$$

$$s^2 = EI/(kAGL^2). \tag{12}$$

From Eqs. (8) and (9), the following expressions can be derived:

$$\frac{\partial^4 Y}{\partial \xi^4} + b^2(r^2 + s^2)\frac{\partial^2 Y}{\partial \xi^2} - b^2(1 - b^2 r^2 s^2) Y = 0, \tag{13}$$

$$\frac{\partial^4 \Psi}{\partial \xi^4} + b^2(r^2 + s^2)\frac{\partial^2 \Psi}{\partial \xi^2} - b^2(1 - b^2 r^2 s^2)\Psi = 0. \tag{14}$$

The general solutions of Eqs. (13) and (14) are

$$Y = C_1\cos b\alpha\xi + C_2\sin b\alpha\xi + C_3\cos b\beta\xi + C_4\sin b\beta\xi, \tag{15}$$

$$\Psi = D_1\sin b\alpha\xi + D_2\cos b\alpha\xi + D_3\sin b\beta\xi + D_4\cos b\beta\xi. \tag{16}$$

In which, $$\frac{\alpha}{\beta} = \left[\frac{r^2 + s^2}{2} \mp \sqrt{\left(\frac{r^2 - s^2}{2}\right)^2 + \frac{1}{b^2}}\right]^{1/2}. \tag{17}$$

Substituting Eqs. (15) ~ (16) into Eq. (9), we have

$$-(C_1 b^2\alpha^2\cos b\alpha\xi + C_2 b^2\alpha^2\sin b\alpha\xi + C_3 b^2\beta^2\cos b\beta\xi + C_4 b^2\beta^2\sin b\beta\xi) + b^2 s^2(C_1\cos b\alpha\xi + C_2\sin b\alpha\xi + C_3\cos b\beta\xi + C_4\sin b\beta\xi) - L(D_1 b\alpha\cos b\alpha\xi - D_2 b\alpha\sin b\alpha\xi + D_3 b\beta\cos b\beta\xi - D_4 b\beta\sin b\beta\xi) = 0. \tag{18}$$

To ensure the above equality, the coefficients C_1 ~ C_4 and D_1 ~ D_4 must satisfy the following relationships:

$$C_1 = L\alpha/b\cdot D_1/(s^2 - \alpha^2), \tag{19}$$

$$C_2 = -L\alpha/b\cdot D_2/(s^2 - \alpha^2), \tag{20}$$

$$C_3 = L\beta/b\cdot D_3/(s^2 - \beta^2), \tag{21}$$

$$C_4 = -L\beta/b\cdot D_4/(s^2 - \beta^2). \tag{22}$$

2 Natural Frequencies and Modes of the Impact System

Suppose a free-free Timoshenko beam is struck transversely by a rigid mass M_0 with the velocity V_0 at the central point of the beam (as shown in Fig. 1). The beam and the rigid body can be seen as a contact-impact system. Considering the symmetry of the system, the left half part of the beam [0, $L/2$] (as shown in Fig. 2) is chosen for the analysis.

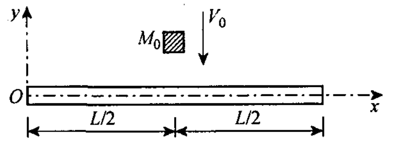

Fig.1 Impact system

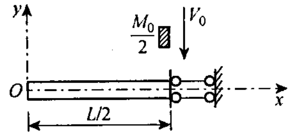

Fig.2 Equivalent impact system

2.1 Natural frequencies

The boundary conditions of the equivalent impact system shown in Fig.2 are as follows:

$$\left(\frac{1}{L}\frac{\partial Y}{\partial \xi} - \Psi\right)\bigg|_{\xi=0} = 0, \tag{23}$$

$$\partial \Psi/\partial \xi\,|_{\xi=0} = 0, \tag{24}$$

$$kAG\left(\frac{1}{L}\frac{\partial Y}{\partial \xi} - \Psi\right)\bigg|_{\xi=1/2} = \frac{1}{L}M_0 \cdot p^2 \cdot Y\bigg|_{\xi=1/2}, \tag{25}$$

$$\Psi\,|_{\xi=1/2} = 0. \tag{26}$$

Substituting Eqs. (15) and (16) into the above boundary conditions, considering Eqs. (19) ~ (22) and introducing the mass ratio $\lambda = \rho \cdot A \cdot L/M_0$, the following equations can be obtained

$$\frac{s^2}{\alpha^2 - s^2}D_2 + \frac{s^2}{\beta^2 - s^2}D_4 = 0, \tag{27}$$

$$b\alpha \cdot D_1 + b\beta \cdot D_3 = 0, \tag{28}$$

$$\begin{aligned}&\frac{L}{b^2(\alpha^2 - s^2)}\left[2\lambda\sin\frac{b\alpha}{2} + b\alpha\cos\frac{b\alpha}{2}\right]D_1 + \frac{L}{b^2(\alpha^2 - s^2)}\left[2\lambda\cos\frac{b\alpha}{2} - b\alpha\sin\frac{b\alpha}{2}\right]D_2 + \\ &\frac{L}{b^2(\beta^2 - s^2)}\left[2\lambda\sin\frac{b\beta}{2} + b\beta\cos\frac{b\beta}{2}\right]D_3 + \\ &\frac{L}{b^2(\beta^2 - s^2)}\left[2\lambda\cos\frac{b\beta}{2} - b\beta\sin\frac{b\beta}{2}\right]D_4 = 0,\end{aligned} \tag{29}$$

$$\sin(b\alpha/2)D_1 + \cos(b\alpha/2)D_2 + \sin(b\beta/2)D_3 + \cos(b\beta/2)D_4 = 0. \tag{30}$$

Equations (27) ~ (30) can be expressed in matrix form

$$\boldsymbol{Ad} = \boldsymbol{0} \tag{31}$$

where

$$\boldsymbol{d} = (D_1, D_2, D_3, D_4)^{\mathrm{T}};$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & s^2/(\alpha^2 - s^2) & 0 & s^2/(\beta^2 - s^2) \\ b\alpha & 0 & b\beta & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ \sin(b\alpha/2) & \cos(b\alpha/2) & \sin(b\beta/2) & \cos(b\beta/2) \end{bmatrix},$$

$$\begin{cases} A_{31} = \dfrac{L(2\lambda\sin(b\alpha/2) + b\alpha\cos(b\alpha/2))}{b^2(\alpha^2 - s^2)};\ A_{32} = \dfrac{L(2\lambda\cos(b\alpha/2) - b\alpha\sin(b\alpha/2))}{b^2(\alpha^2 - s^2)}; \\ A_{33} = \dfrac{L(2\lambda\sin(b\beta/2) + b\beta\cos(b\beta/2))}{b^2(\beta^2 - s^2)};\ A_{34} = \dfrac{L(2\lambda\cos(b\beta/2) - b\beta\sin(b\beta/2))}{b^2(\beta^2 - s^2)}. \end{cases} \tag{32}$$

Only when $|A| = 0$, the nontrivial solutions of the matrix equation (31) exist. Let $f(b) = |A|$, then

$$f(b) = \frac{2L\lambda\cdot\alpha\cdot s^2}{b(\alpha^2 - s^2)^2}\left(1 - \frac{\alpha^2 - s^2}{\beta^2 - s^2}\right)\left(\frac{\beta}{\alpha}\sin\frac{b\alpha}{2}\cos\frac{b\beta}{2} - \frac{\alpha^2 - s^2}{\beta^2 - s^2}\sin\frac{b\beta}{2}\cos\frac{b\alpha}{2}\right) + \frac{L\alpha\cdot\beta\cdot s^2}{(\alpha^2 - s^2)(\beta^2 - s^2)}\left[-2 + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)\sin\frac{b\alpha}{2}\sin\frac{b\beta}{2} + \left(\frac{\alpha^2 - s^2}{\beta^2 - s^2} + \frac{\beta^2 - s^2}{\alpha^2 - s^2}\right)\cos\frac{b\alpha}{2}\cos\frac{b\beta}{2}\right] = 0. \tag{33}$$

Equation (33) is just the characteristic equation of the impact system. It has unlimited positive real roots b_n, $n = 1,2,3,\cdots$, which correspond to the infinite natural frequencies of the system p_n, $n = 1,2,3,\cdots$ (corresponding relationship is Eq.(10)).

2.2 Natural modes

From Eqs.(27) and (28), following expressions can be derived

$$\zeta = \frac{D_4}{D_2} = -\frac{\beta^2 - s^2}{\alpha^2 - s^2}, \tag{34}$$

$$\mu = \frac{D_3}{D_1} = -\frac{\alpha}{\beta}. \tag{35}$$

From Eqs.(27),(28) and (30), we have

$$\eta = \frac{D_2}{D_1} = -\frac{\sin\frac{b\alpha}{2} + \mu\cdot\sin\frac{b\beta}{2}}{\cos\frac{b\alpha}{2} + \zeta\cdot\cos\frac{b\beta}{2}}. \tag{36}$$

Substituting Eqs.(34) ~ (36) into Eq.(16), we have

$$\Psi = D_1\cdot(\sin b\alpha\xi + \eta\cos b\alpha\xi + \mu\sin b\beta\xi + \eta\zeta\cos b\beta\xi). \tag{37}$$

Substituting Eqs.(19) ~ (22) into Eq.(15), and combining Eqs.(34) ~ (36) gives

$$Y = \chi\cdot D_1\cdot\left(\cos b\alpha\xi - \eta\sin b\alpha\xi + \frac{1}{\zeta}\cos b\beta\xi - \frac{\eta}{\mu}\sin b\beta\xi\right). \tag{38}$$

In which,

$$\chi = \frac{L\alpha}{b(s^2 - \alpha^2)}. \tag{39}$$

For every real root b_n, $n = 1,2,3,\cdots$ of the characteristic equation (33), there are the corresponding modes $Y_n(\xi)$ and $\Psi_n(\xi)$ of the impact system:

$$Y_n = \chi_n\cdot\left(\cos b_n\alpha_n\xi - \eta_n\sin b_n\alpha_n\xi + \frac{1}{\zeta_n}\cos b_n\beta_n\xi - \frac{\eta_n}{\mu_n}\sin b_n\beta_n\xi\right), \tag{40}$$

$$\Psi_n = \sin b_n\alpha_n\xi + \eta_n\cos b_n\alpha_n\xi + \mu_n\sin b_n\beta_n\xi + \eta_n\zeta_n\cos b_n\beta_n\xi). \tag{41}$$

3 Dynamic Responses of the Impact System

From Eqs.(37) and (38), it can be seen that there is one-to-one correspondence between every order of modes $Y_n(\xi)$ and $\Psi_n(\xi)$; in addition, because of the symmetry of the structure and the loading, the beam has only rigid translation displacement, but has no rigid rotation displacement. So the transverse displacement and bending rotation angle of the impact system can be written as follows:

$$y(\xi,t) = A_0\cdot t + \sum_{n=1}^{+\infty} A_n\cdot Y_n(\xi)\cdot \sin p_n t, \tag{42}$$

$$\psi(\xi,t) = \sum_{n=1}^{+\infty} A_n\cdot \Psi_n(\xi)\cdot \sin p_n t. \tag{43}$$

The first term of right-hand side in Eq.(42) represents the rigid response, that is to say, the zero frequency response of the impact system, whose mode is $Y_0(\xi) = 1$. The series terms in Eqs.(42) and (43) represent the elastic responses of the impact system. The former is the deflection of the beam, and the latter is the bending angle of rotation of the beam cross-section, respectively. In calculation, enough orders of natural frequencies should be chosen to satisfy the requried precision for various responses.

The initial conditions of the impact system are

$$\left.\frac{\partial y(\xi,t)}{\partial t}\right|_{t=0} = \begin{cases} -V_0 & (\xi = 1/2), \\ 0 & (0 \leqslant \xi \leqslant 1/2), \end{cases} \tag{44}$$

$$\left.\frac{\partial \psi(\xi,t)}{\partial t}\right|_{t=0} = 0. \tag{45}$$

From Eqs.(42) and (43), following expressions can be derived

$$g(\xi) = \left.\frac{\partial y(\xi,t)}{\partial t}\right|_{t=0} = A_0 + \sum_{n=1}^{+\infty} A_n\cdot p_n\cdot Y_n(\xi), \tag{46}$$

$$h(\xi) = \left.\frac{\partial \psi(\xi,t)}{\partial t}\right|_{t=0} = \sum A_n\cdot p_n\cdot \Psi_n(\xi). \tag{47}$$

3.1 The factor A_0 of rigid response

Multiplying both sides of Eq.(46) by $Y_0(\xi) = 1$ and the distributed mass of the beam $\bar{m}(\xi) = \rho AL, 0 \leqslant \xi \leqslant 1/2$, considering the concentrate rigid mass $\bar{M} = M_0/2$ at the impact point $\xi = 1/2$, and integrating them with respect to ξ over the length of the beam $[0,1/2)$, we have

$$\lim_{\varepsilon\to+0}\left[\int_0^{1/2-\varepsilon} \bar{m}(\xi)\cdot Y_0(\xi)\cdot g(\xi)\cdot d\xi\right] + \bar{M}\cdot Y_0\left(\frac{1}{2}\right)\cdot g\left(\frac{1}{2}\right) =$$
$$A_0\left\{\lim_{\varepsilon\to+0}\left[\int_0^{1/2-\varepsilon} \bar{m}(\xi)\cdot Y_0(\xi)\cdot d\xi\right] + \bar{M}\cdot Y_0\left(\frac{1}{2}\right)\right\} +$$
$$\sum_{n=1}^{+\infty} A_n\cdot p_n\cdot\left\{\lim_{\varepsilon\to+0}\left[\int_0^{1/2-\varepsilon} \bar{m}(\xi)\cdot Y_0(\xi)\cdot Y_n(\xi)\cdot d\xi\right] +\right.$$
$$\left.\bar{M}\cdot Y_0\left(\frac{1}{2}\right)\cdot Y_n\left(\frac{1}{2}\right)\right\}. \tag{48}$$

For the rigid mode $Y_0(\xi) = 1$ and the elastic mode $Y_n(\xi)$, the orthogonality condition can be found as

$$\lim_{\varepsilon\to+0}\left[\int_0^{1/2-\varepsilon} \bar{m}(\xi)\cdot Y_0(\xi)\cdot Y_n(\xi)\cdot d\xi\right] + \bar{M}\cdot Y_0\left(\frac{1}{2}\right)\cdot Y_n\left(\frac{1}{2}\right) = 0 \qquad (n = 1,2,3,\cdots). \tag{49}$$

Substituting the initial conditions (44) ~ (47) and the orthogonality condition (49) into Eq.(48) gives

$$A_0 = \frac{1}{1+\lambda}\cdot(-V_0). \tag{50}$$

The factor A_0 is just the rigid velocity of the system $v_r(v_r = (-V_0)/(1+\lambda))$. So the momentum of rigid response for the system is $(1+\lambda)\cdot M_0\cdot(-V_0)/(1+\lambda) = -M_0\cdot V_0$ which equals to the initial momentum of the system before the contact-impact process. According to the principle of momentum conservation, the momentum sum of elastic responses for the system should be zero. From this view point, the rigid response of the system can be evaluated directly.

3.2 The factor A_n of elastic response

Multiplying both sides of Eq.(46) by $Y_m(\xi)$ and the distributed mass of the beam $\bar{m}(\xi) = \rho AL, 0 \leqslant \xi < 1/2$, considering the concentrate rigid mass $\bar{M} = M_0/2$ at the impact point $\xi = 1/2$, and integrating them with respect to ξ over the length of the beam $[0,1/2)$, we have

$$\begin{aligned}&\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon}\bar{m}(\xi)\cdot Y_m(\xi)\cdot g(\xi)\cdot d\xi\Big] + \bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\cdot g\Big(\frac{1}{2}\Big) = \\ &A_0\Big\{\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon}\bar{m}(\xi)\cdot Y_m(\xi)\cdot d\xi\Big] + \bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\Big\} + \\ &\sum_{n=1}^{+\infty}A_n\cdot p_n\cdot\Big\{\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon}\bar{m}(\xi)\cdot Y_m(\xi)\cdot Y_n(\xi)\cdot d\xi\Big] + \\ &\bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\cdot Y_n\Big(\frac{1}{2}\Big)\Big\}.\end{aligned} \tag{51}$$

Multiplying both sides of Eq.(47) by $\Psi_m(\xi)$ and the distributed rotation inertia of the beam cross-section $\bar{I}(\xi) = \rho IL, 0 \leqslant \xi < 1/2$, considering the bending angle of rotation $\Psi_m(1/2) = 0$ at the impact point $\xi = 1/2$, and integrating them with respect to ξ over the length of the beam $[0,1/2)$, we have

$$\begin{aligned}&\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon}\bar{I}(\xi)\cdot\Psi_m(\xi)\cdot h(\xi)\cdot d\xi\Big] = \\ &\sum_{n=1}^{+\infty}A_n\cdot p_n\cdot\Big\{\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon}\bar{I}(\xi)\cdot\Psi_m(\xi)\cdot\Psi_n(\xi)\cdot d\xi\Big]\Big\}.\end{aligned} \tag{52}$$

Superposing Eq.(51) and Eq.(52), combining the initial conditions (44) ~ (47) and the orthogonality condition (49), we can get

$$\begin{aligned}&\bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\cdot g\Big(\frac{1}{2}\Big) = \sum_{n=1}^{+\infty}A_n\cdot p_n\times \\ &\Big\{\lim_{\varepsilon\to+0}\Big\{\int_0^{1/2-\varepsilon}[\bar{m}(\xi)\cdot Y_m(\xi)\cdot Y_n(\xi) + \bar{I}(\xi)\cdot\Psi_m(\xi)\cdot\Psi_n(\xi)]\cdot d\xi\Big\} + \\ &\bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\cdot Y_n\Big(\frac{1}{2}\Big)\Big\}.\end{aligned} \tag{53}$$

For the transverse displacement mode $Y_n(\xi)$ and the rotation angle mode $\Psi_n(\xi)$, the orthogonality condition can be written as

$$\begin{aligned}&\lim_{\varepsilon\to+0}\Big\{\int_0^{1/2-\varepsilon}[\bar{m}(\xi)\cdot Y_m(\xi)\cdot Y_n(\xi) + \bar{I}(\xi)\cdot\Psi_m(\xi)\cdot\Psi_n(\xi)]\cdot d\xi\Big\} + \\ &\bar{M}\cdot Y_m\Big(\frac{1}{2}\Big)\cdot Y_n\Big(\frac{1}{2}\Big) = 0 \qquad (m \neq n).\end{aligned} \tag{54}$$

Substituting Eq.(54) into Eq.(53), the general coefficient A_n can be obtained as

$$A_n = \frac{\bar{M}\cdot Y_n\left(\frac{1}{2}\right)\cdot g\left(\frac{1}{2}\right)}{p_n\cdot\left\{\lim_{\varepsilon\to+0}\left\{\int_0^{1/2-\varepsilon}\left[\bar{m}(\xi)\cdot Y_n^2(\xi)+\bar{I}(\xi)\cdot\Psi^2(\xi)\right]\cdot d\xi\right\}+\bar{M}\cdot Y_n^2\left(\frac{1}{2}\right)\right\}}. \tag{55}$$

With the flexural displacement and the rotation angle responses known, the dynamic responses of the beam, such as the velocity $V(\xi,t)$, the shear force $Q(\xi,t)$ and the impact force $P(t)$, etc., can be derived as follows:

$$V(\xi,t) = \frac{\partial y(\xi,t)}{\partial t} = A_0 + \sum_{n=1}^{+\infty} A_n\cdot p_n\cdot Y_n(\xi)\cdot\cos p_n t, \tag{56}$$

$$Q(\xi,t) = -kAG\cdot\left[\frac{1}{L}\cdot\frac{\partial y(\xi,t)}{\partial t} - \psi(\xi,t)\right] =$$

$$-kAG\cdot\sum_{n=1}^{+\infty} A_n\cdot\left[\frac{1}{L}\cdot\frac{\partial Y_n(\xi)}{\partial\xi} - \Psi_n(\xi)\right]\cdot\sin p_n t. \tag{57}$$

Because of the symmetry of the structure and the loading, the shear force distributes antisymmetrically about the center of the beam. So the shear force of the beam is discontinued at the impact point ($\xi = 1/2$). The shear forces at both sides of the point are $Q(1/2-0,t)$ and $Q(1/2+0,t)$, respectively, where $(1/2-0)$ and $(1/2+0)$ represent the left side and right side of $\xi = 1/2$, respectively. Therefore, the impact force of the moving rigid body is

$$P(t) = Q\left(\frac{1}{2}+1,t\right) - Q\left(\frac{1}{2}-0,t\right) = -2Q\left(\frac{1}{2}-0,t\right). \tag{58}$$

4 Numerical Example

The parameters of example beam are: the length of the beam $L = 36.576$ cm; area of its cross-section $A = 6.452\ \text{cm}^2$; its mass density $\rho = 7.75\times10^3\ \text{kg/m}^3$; the shape factor of its cross-section $k = 2/3$; the modulus ratio of elastic to shear $E/G = 8/3$; the mass ratio of beam to rigid body $\lambda = 1$; from Eqs. (11) and (12) it can be found that $s = 2r$; if $r = 0.02$, then $s = 0.04$. The wave speeds of flexural wave front and shear wave in the beam can be evaluated, which are $c = \sqrt{E/\rho}$ (i.e., the longitudinal wave speed) and $c_s = \sqrt{kG/\rho}$, respectively, and the ratio of them is $c/c_s = 2$.

Figure 3 shows the dimensionless impact force between the beam and the rigid body at the impact end of the beam. From the figure it can be seen that from the dimensionless time $\tau = tc/L = 1$, i.e., the time used up by the flexural wave propagating to the free end ($\xi = 0$) and by its reflective wave from the end arriving at the impact end ($\xi = 1/2$), the impact force starts to oscillate slightly. From the time $\tau = 1.5$, i.e., the time when the wave returns again to the free end, the force oscillates evidently. At about the time $\tau = 1.6$, the impact force diminishes to zero, that means the impact rigid body begins to separate from the beam, and the impact process finishes.

Figure 4 shows the shear force distribution along the beam at the time $\tau = 0.4$. From this figure it can be found that in the regions $\xi = 0 \sim 0.1$ and $\xi = 0.9 \sim 1$, the shear forces are zero, this is because the flexural wave front still does not arrive in these regions. From the figure it can also be seen that the shear force has sharp peak values at the points $\xi = 0.3$ and $\xi = 0.7$, which dues to that the shear waves just reach these points at the moment $\tau = 0.4$. Since the shear force exhibits an antisymmetrical distribution along the length of the beam about the beam center, the

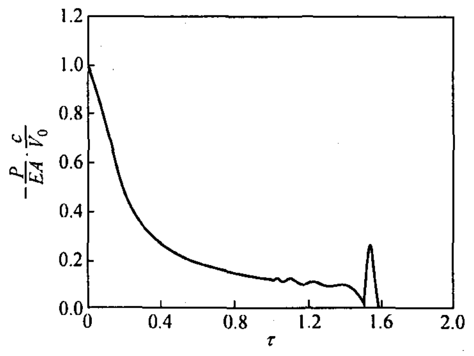

Fig.3 Impact force

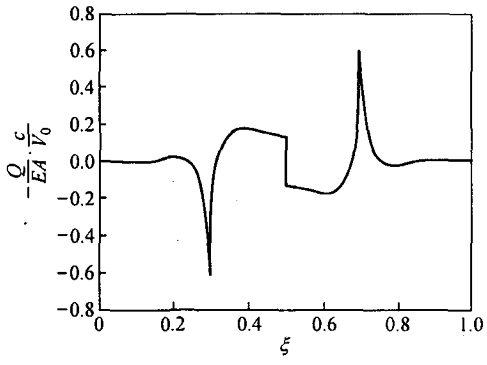

Fig.4 Shear distribution (τ = 0.4)

shear force jumps at the point ξ = 1/2.

Compared with Ref.[7], it can be found that the elastic response characteristics of impacted unrestrained beam are similar to those of simply supported beam.

Figure 5 and Fig.6 show the dimensionless velocity distribution along the beam length at the time τ = 0.4. The horizontal straight line in Fig.5 denotes the rigid velocity of the beam, i.e., V_r which is just the constant term A_0 of the right-hand side in Eq.(56). The mass ratio is taken as λ = 1 in the example, so the rigid velocity of the beam is $V_r = 0.5(-V_0)$ according to Eq.(50). The curved line in Fig.5 represents the elastic velocity distribution of the beam, i.e., V_e which is the serial part of Eq.(56). Fig.6 represents the total velocity distribution of the beam, i.e., V which is the summation of the rigid velocity V_r and the elastic velocity V_e. It can be seen that the distribution characteristics of velocity is the same as that of shear force except for the symmetry of velocity distribution.

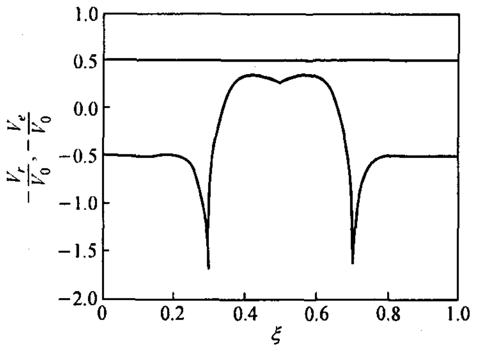

Fig. 5 Distributions of rigid velocity and elastic velocity (τ = 0.4)

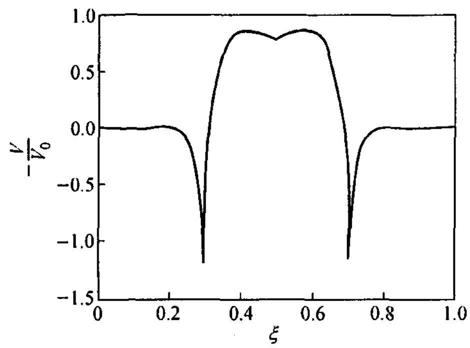

Fig.6 Velocity distribution (τ = 0.4)

5 Conclusions

1) The dynamic responses of an unrestrained Timoshenko beam to transverse impact of a rigid body at its center are composed of two parts: rigid responses and elastic responses. The numerical example reveals that the elastic responses in the total impact responses of the unrestrained Timoshenko beam are nearly the same as the impact responses of the simply

supported beam.

2) The momentum sum of the elastic responses in the impact system is always zero. According to the principle of the momentum conservation, the momentum of the rigid responses in the impact system equals to that of the moving rigid body before impact, which makes the rigid responses of the system easy to evaluate. The rigid responses of the system mainly depend on the mass ratio of the unrestrained beam to the moving rigid body.

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