Applied Mathematics and Mechanics
(English Edition) Vol. 26 No. 11 Nov. 2005

Published by Shanghai University,
Shanghai, China

 ISSN 0253-4827 **Article ID**: 0253-4827(2005)11-1449-09

ANALYSIS ON IMPACT RESPONSES OF UNRESTRAINED PLANAR FRAME STRUCTURE(Ⅰ)—FORMULA DERIVATION *

CHEN Rong (陈 镕) [1,2], ZHENG Hai-tao (郑海涛) [1],
XUE Song-tao (薛松涛) [1,2,3], TANG He-sheng (唐和生) [1,2]

(1. Research Institution of Structural Engineering and Disaster Reduction,
Tongji University, Shanghai 200092, P. R. China;
2. Center for Aerospace Technology, Tongji University,
Shanghai 200092, P. R. China;
3. Department of Architecture, School of Science and Engineering,
Kinki University, Osaka 577-8502, Japan)

(Communicated by HE Fu-bao, Original Member of Editorial Committee, AMM)

Abstract: The generalized Fourier-series method was used to derive the impact responses formula of an unrestrained planar frame structure when subjected to an impact of a moving rigid-body. By using these formula, the analytic solutions of dynamic responses of the contact-impact system can be obtained. During the derivation, the momentum sum of elastic responses of the contact-impact system is demonstrated to be zero. From the derivation, it is seen that the modal method can also be used to solve this kind of impact problem.

Key words: unrestrained planar frame structure; impact; rigid response; elastic response

Chinese Library Classification: O347.1 **Document code**: A
2000 Mathematics Subject Classification: 74M20

Introduction

Recently, there have been considerable studies on transient responses and wave propagation of restrained structures which are being impacted[1,2]. But, for the unrestrained structures, there are few relative literatures having been published. Unrestrained structures are mainly space structures, which often are very important structures, for examples: space vehicles, space stations and satellites which are moving in the small-gravity or zero-gravity environment. When these structures are subjected to impacts of moving bodies, even if slight impacts, the motion of these structures or the experiment task executed in these structures may be affected severely so much as to lead to failure of the experiment.

---

* Received May 19, 2003; Revised Jul. 01, 2005
Project supported by the National Science Fund for Distinguished Young Scholars (No. 59925820)
Corresponding author CHEN Rong, Professor, Doctor, E-mail: chenrong@mail.tongji.edu.cn

Therefore, impact responses of these types of structures need to be analyzed precisely. Moreover, it is hoped that this investigation will be useful for spacecraft docking and for vibration control of space stations or satellites.

We have studied the problems of longitudinal impact for an unrestrained bar and transverse impact for an unrestrained Timoshenko beam[3], obtained the analytic solutions of impact responses of these impact systems, and concluded that the momentum sum of elastic responses of the impact system is equal to zero. For the impact problem of the unrestrained structure which is struck by a moving rigid-body, whether the same method (the general Fourier-series method) can be used to solve the impact responses? Whether the impact system follows the same regularity? And whether the calculation results can illuminate clearly the propagation phenomena of stress waves in the structural members? They arouse our great interest. With these questions, in this paper we analyze an unrestrained planar structure which is subjected to the impact of a moving rigid-body. The paper includes two parts. In Part I, the instant response formulas of an unrestrained planar frame structure subjected to the impact of a moving rigid body are derived. In Part Ⅱ, the numerical example is given.

1 Impact System

A symmetry unrestrained planar frame structure (as shown in Fig. 1) is considered. The length of the frame is L and its width is $L/2$. The frame structure is composed of two square spans, each side-length of which is $L/2$. The structure is struck by a rigid mass M_0 with the velocity S_0 at the joint B along its symmetry axis. In the structure, the joints A, C, D and F are pin-connected joints, and the joints B, E are rigidly-connected joints. Since the structure members AD, CF are pin-connected at each end of them, they can be seen as bar elements. Due to the symmetry of the structure and loading, there are only axial deformations but no transverse deformations for member BE, so it can also be seen as bar element. The members AC, DF are both analyzed by Timoshenko Theory of beams, considering the effect of transverse shear deformation.

The moving rigid-body and the unrestrained structure are considered as a contact-impact system. Considering the symmetry of the system, half part of it (as shown in Fig. 2) is chosen for analysis. Here, we denote beam AB and beam DE by beam 1 and beam 2, bar DA and bar EB bar by bar 3 and bar 4, respectively. For the members 1, 2, 3 and 4, we

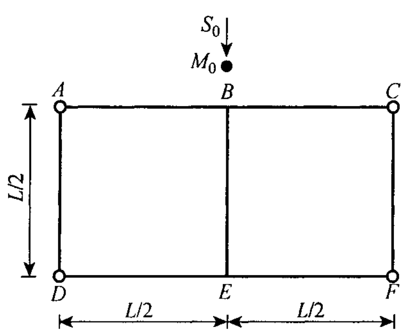

Fig. 1 Impact system

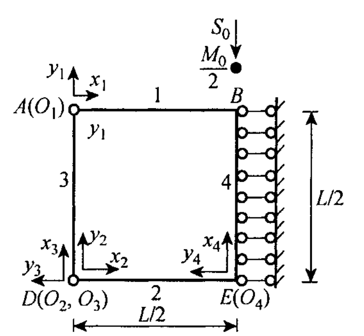

Fig. 2 Equivalent impact system

introduce the local coordinate systems (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), with the corresponding coordinate origins of O_1, O_2, O_3 and O_4 at the joints A, D, D and E, respectively.

2 Modal Functions

2.1 Modal functions of Timoshenko beam

The motion equations for a Timoshenko beam are

$$EI\frac{\partial^2\psi}{\partial x^2} + kAG\left(\frac{\partial v}{\partial x} - \psi\right) - \rho I\frac{\partial^2\psi}{\partial t^2} = 0, \tag{1}$$

$$\rho A\frac{\partial^2 v}{\partial t^2} - kAG\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial\psi}{\partial x}\right) = 0, \tag{2}$$

where $v(x,t)$ is the transverse displacement of the beam; $\psi(x,t)$ is the bending rotation angle of the beam's cross-section; E, G, I, A and ρ are the elastic modulus, shear modulus, inertia moment of the cross-section, cross-sectional area and mass density of the beam, respectively; k is the shape factor of the cross-section.

The transverse displacements and bending rotation angles of beams 1 and 2 can be written as

$$v_i(\xi_i, t) = V_i(\xi_i, p)\cdot\sin pt, \tag{3}$$

$$\psi_i(\xi_i, t) = \Psi_i(\xi_i, p)\cdot\sin pt, \tag{4}$$

in which $i = 1, 2$, denoting beams 1 and 2, respectively; $\xi_i = x_i/L$ are the dimensionless coordinates along the axis of beams 1 and 2; p is the circular frequency of the impact system; L is the length of the rectangle frame.

Through derivation, the modal functions of transverse displacements and bending rotations of beams 1 and 2 can be written as the following forms[4]:

$$V_i(\xi_i) = C_{i1}\cdot\cos(b_i\alpha_i\xi_i) + C_{i2}\cdot\sin(b_i\alpha_i\xi_i) + C_{i3}\cdot\cos(b_i\beta_i\xi_i) + C_{i4}\cdot\sin(b_i\beta_i\xi_i), \tag{5}$$

$$\Psi_i(\xi_i) = D_{i1}\cdot\sin(b_i\alpha_i\xi_i) + D_{i2}\cdot\cos(b_i\alpha_i\xi_i) + D_{i3}\cdot\sin(b_i\beta_i\xi_i) + D_{i4}\cdot\cos(b_i\beta_i\xi_i), \tag{6}$$

in which

$$\begin{matrix}\alpha_i\\ \beta_i\end{matrix} = \left[\frac{r_i^2 + s_i^2}{2} \mp \sqrt{\left(\frac{r_i^2 - s_i^2}{2}\right)^2 + \frac{1}{b_i^2}}\right]^{1/2}, \quad b_i^2 = \frac{1}{E_i I_i}\rho_i A_i L^4 p^2,\ r_i^2 = \frac{I_i}{A_i L^2},\ s_i^2 = \frac{E_i I_i}{k_i A_i G_i L^2}.$$

It can be derived further that the coefficients $C_{i1} \sim C_{i4}$ and $D_{i1} \sim D_{i4}$ in Eq. (5) and Eq. (6) must satisfy the following relationships[4]:

$$C_{i1} = \frac{L\,\alpha_i}{b_i}\cdot\frac{1}{s_i^2 - \alpha_i^2}D_{i1}, \quad C_{i2} = \frac{L\,\alpha_i}{b_i}\cdot\frac{1}{s_i^2 - \alpha_i^2}D_{i2},$$

$$C_{i3} = \frac{L\,\beta_i}{b_i}\cdot\frac{1}{s_i^2 - \beta_i^2}D_{i3}, \quad C_{i4} = \frac{L\,\beta_i}{b_i}\cdot\frac{1}{s_i^2 - \beta_i^2}D_{i4}.$$

2.2 Modal functions of the bars

The motion equation for a bar can be written as:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\rho}{E}\cdot\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{7}$$

The longitudinal displacements of bars 3 and 4 are given by

$$u_j(x_j, t) = U_j(\xi_j, p)\cdot\sin pt, \tag{8}$$

in which $j = 3, 4$, denoting bars 3 and 4, respectively; The definitions of ξ_j and p are the same as before.

Substituting Eq. (8) into Eq. (7), the modal functions of longitudinal displacements of bars 3 and 4 are obtained

$$U_j(\xi_j) = D_{j1}\cdot\cos(\Omega_j\xi_j) + D_{j2}\cdot\sin(\Omega_j\xi_j), \tag{9}$$

in which $\Omega_j = b_j\cdot r_j$, $b_j^2 = \rho_j A_j L^4 p^2/E_j I_j$, $r_j^2 = I_j/A_j L^2$.

3 Boundary Conditions

For the equivalent impact system as shown in Fig. 2, the conditions of equilibrium and compatibility at each joints of the structure can be separately expressed as follows:

The condition of equilibrium at joint A:

$$k_1 A_1 G_1\cdot\left(\frac{1}{L}\cdot\frac{\partial V_1}{\partial\xi_1} - \Psi_1\right)\bigg|_{\xi_1=0} = E_3 A_3\cdot\frac{1}{L}\cdot\frac{\partial U_3}{\partial\xi_3}\bigg|_{\xi_3=1/2}, \quad \frac{\partial\Psi_1}{\partial\xi_1}\bigg|_{\xi_1=0} = 0. \tag{10}$$

The condition of equilibrium at joint B:

$$k_1 A_1 G_1\left(\frac{1}{L}\cdot\frac{\partial V_1}{\partial\xi_1} - \Psi_1\right)\bigg|_{\xi_1=1/2} - \frac{1}{2}\cdot M_0\cdot p^2\cdot V_1\bigg|_{\xi_1=1/2} = -E_4\cdot\frac{A_4}{2}\cdot\frac{1}{L}\cdot\frac{\partial U_4}{\partial\xi_4}\bigg|_{\xi_4=1/2}, \quad \Psi_1\bigg|_{\xi_1=1/2} = 0. \tag{11}$$

The condition of equilibrium at joint D:

$$k_2 A_2 G_2\cdot\left(\frac{1}{L}\cdot\frac{\partial V_2}{\partial\xi_2} - \Psi_2\right)\bigg|_{\xi_2=0} = -E_3 A_3\cdot\frac{1}{L}\cdot\frac{\partial U_3}{\partial\xi_3}\bigg|_{\xi_3=0}, \quad \frac{\partial\Psi_2}{\partial\xi_2}\bigg|_{\xi_2=0} = 0. \tag{12}$$

The condition of equilibrium at joint E:

$$k_2 A_2 G_2\cdot\left(\frac{1}{L}\cdot\frac{\partial V_2}{\partial\xi_2} - \Psi_2\right)\bigg|_{\xi_2=1/2} = E_4\cdot\frac{A_4}{2}\cdot\frac{1}{L}\cdot\frac{\partial U_4}{\partial\xi_4}\bigg|_{\xi_4=0}, \quad \Psi_2\bigg|_{\xi_2=1/2} = 0. \tag{13}$$

The condition of compatibility at joint A:

$$V_1\big|_{\xi_1=0} = U_3\big|_{\xi_3=1/2}. \tag{14}$$

The condition of compatibility at joint B:

$$V_1\big|_{\xi_1=1/2} = U_4\big|_{\xi_4=1/2}. \tag{15}$$

The condition of compatibility at joint D:

$$V_2\big|_{\xi_2=0} = U_3\big|_{\xi_3=0}. \tag{16}$$

The condition of compatibility at joint E:

$$V_2\big|_{\xi_2=1/2} = U_4\big|_{\xi_4=0}. \tag{17}$$

4 Characteristic Equations and Natural Frequencies

Substituting Eq. (5), Eq. (6) and Eq. (9) into the above boundary conditions Eqs. (10) - (17), considering the relations of C_{ij} and $D_{ij}(i, j = 1, \cdots, 4)$, and introducing the mass ratio of beam AC to the moving rigid-body $\lambda = \rho_1 A_1 L/M_0$, the homogeneous linear equations set with respect to the coefficients in the modal functions can be obtained

$$\frac{s_1^2}{\alpha_1^2 - s_1^2}\cdot D_{12} + \frac{s_1^2}{\beta_1^2 - s_1^2}\cdot D_{14} + \frac{E_3 A_3\cdot b_3 r_3}{k_1 A_1 G_1\cdot L}\cdot\left(\sin\frac{b_3 r_3}{2}\cdot D_{31} - \cos\frac{b_3 r_3}{2}\cdot D_{32}\right) = 0, \tag{18}$$

$$b_1\alpha_1\cdot D_{11} + b_1\beta_1\cdot D_{13} = 0, \tag{19}$$

$$\frac{s_1^2}{\alpha_1^2 - s_1^2}\left(\sin\frac{b_1\alpha_1}{2} + \frac{1}{2}\cdot\frac{b_1\alpha_1}{\lambda}\cdot\cos\frac{b_1\alpha_1}{2}\right)\cdot D_{11} + \frac{s_1^2}{\alpha_1^2 - s_1^2}\left(\cos\frac{b_1\alpha_1}{2} - \frac{1}{2}\cdot\frac{b_1\alpha_1}{\lambda}\cdot\sin\frac{b_1\alpha_1}{2}\right)\cdot D_{12}$$

$$+ \frac{s_1^2}{\beta_1^2 - s_1^2}\left(\sin\frac{b_1\beta_1}{2} + \frac{1}{2}\cdot\frac{b_1\beta_1}{\lambda}\cdot\cos\frac{b_1\beta_1}{2}\right)\cdot D_{13} + \frac{s_1^2}{\beta_1^2 - s_1^2}\left(\cos\frac{b_1\beta_1}{2} - \frac{1}{2}\cdot\frac{b_1\beta_1}{\lambda}\cdot\sin\frac{b_1\beta_1}{2}\right)\cdot D_{14}$$

$$- \frac{1}{2}\cdot\frac{E_4A_4\cdot b_4r_4}{k_1A_1G_1\cdot L}\cdot\left(\sin\frac{b_4r_4}{2}\cdot D_{41} - \cos\frac{b_4r_4}{2}\cdot D_{42}\right) = 0, \tag{20}$$

$$\sin\frac{b_1\alpha_1}{2}\cdot D_{11} + \cos\frac{b_1\alpha_1}{2}\cdot D_{12} + \sin\frac{b_1\beta_1}{2}\cdot D_{13} + \cos\frac{b_1\beta_1}{2}\cdot D_{14} = 0, \tag{21}$$

$$\frac{s_2^2}{\alpha_2^2 - s_2^2}\cdot D_{22} + \frac{s_2^2}{\beta_2^2 - s_2^2}\cdot D_{24} + \frac{E_3A_3\cdot b_3r_3}{k_2A_2G_2\cdot L}\cdot D_{32} = 0, \tag{22}$$

$$b_2\alpha_2\cdot D_{21} + b_2\beta_2\cdot D_{23} = 0, \tag{23}$$

$$\frac{s_2^2}{\alpha_2^2 - s_2^2}\cdot\sin\frac{b_2\alpha_2}{2}\cdot D_{21} + \frac{s_2^2}{\alpha_2^2 - s_2^2}\cdot\cos\frac{b_2\alpha_2}{2}\cdot D_{22} + \frac{s_2^2}{\beta_2^2 - s_2^2}\cdot\sin\frac{b_2\beta_2}{2}\cdot D_{23}$$

$$+ \frac{s_2^2}{\beta_2^2 - s_2^2}\cdot\cos\frac{b_2\beta_2}{2}\cdot D_{24} - \frac{1}{2}\cdot\frac{E_4A_4\cdot b_4r_4}{k_2A_2G_2\cdot L}\cdot D_{42} = 0, \tag{24}$$

$$\sin\frac{b_2\alpha_2}{2}\cdot D_{21} + \cos\frac{b_2\alpha_2}{2}\cdot D_{22} + \sin\frac{b_2\beta_2}{2}\cdot D_{23} + \cos\frac{b_2\beta_2}{2}\cdot D_{24} = 0, \tag{25}$$

$$\frac{L\,\alpha_1}{b_1}\cdot\frac{1}{\alpha_1^2 - s_1^2}\cdot D_{11} + \frac{L\,\beta_1}{b_1}\cdot\frac{1}{\beta_1^2 - s_1^2}\cdot D_{13} + \cos\frac{b_3r_3}{2}\cdot D_{31} + \sin\frac{b_3r_3}{2}\cdot D_{32} = 0, \tag{26}$$

$$\frac{L\,\alpha_1}{b_1}\cdot\frac{1}{\alpha_1^2 - s_1^2}\cdot\cos\frac{b_1\alpha_1}{2}\cdot D_{11} - \frac{L\,\alpha_1}{b_1}\cdot\frac{1}{\alpha_1^2 - s_1^2}\cdot\sin\frac{b_1\alpha_1}{2}\cdot D_{12} + \frac{L\,\beta_1}{b_1}\cdot\frac{1}{\beta_1^2 - s_1^2}\cdot\cos\frac{b_1\beta_1}{2}\cdot D_{13}$$

$$- \frac{L\,\beta_1}{b_1}\cdot\frac{1}{\beta_1^2 - s_1^2}\cdot\sin\frac{b_1\beta_1}{2}\cdot D_{14} + \cos\frac{b_4r_4}{2}\cdot D_{41} + \sin\frac{b_4r_4}{2}\cdot D_{42} = 0, \tag{27}$$

$$\frac{L\,\alpha_2}{b_2}\cdot\frac{1}{\alpha_2^2 - s_2^2}\cdot D_{21} + \frac{L\,\beta_2}{b_2}\cdot\frac{1}{\beta_2^2 - s_2^2}\cdot D_{23} + D_{31} = 0, \tag{28}$$

$$\frac{L\,\alpha_2}{b_2}\cdot\frac{1}{\alpha_2^2 - s_2^2}\cdot\cos\frac{b_2\alpha_2}{2}\cdot D_{21} - \frac{L\,\alpha_2}{b_2}\cdot\frac{1}{\alpha_2^2 - s_2^2}\cdot\sin\frac{b_2\alpha_2}{2}\cdot D_{22}$$

$$+ \frac{L\,\beta_2}{b_2}\cdot\frac{1}{\beta_2^2 - s_2^2}\cdot\cos\frac{b_2\beta_2}{2}\cdot D_{23} - \frac{L\,\beta_2}{b_2}\cdot\frac{1}{\beta_2^2 - s_2^2}\cdot\sin\frac{b_2\beta_2}{2}\cdot D_{24} + D_{41} = 0. \tag{29}$$

Equations (18) – (29) can be written in matrix form:

$$\hat{A}\cdot\boldsymbol{d} = \boldsymbol{0}, \tag{30}$$

in which $\boldsymbol{d} = (D_{11}, D_{12}, D_{13}, D_{14}, D_{21}, D_{22}, D_{23}, D_{24}, D_{31}, D_{32}, D_{41}, D_{42})^{\mathrm{T}}$ is the vector composed of coefficients in displacement and angular modal functions $V_1(\xi_1), \Psi_1(\xi_1), V_2(\xi_2), \Psi_2(\xi_2), U_3(\xi_3)$ and $U_4(\xi_4)$.

The parameters are not the same for different members, so the dimensionless frequencies $b_k, k = 1, \cdots, 4$ for different members differ from each other even though the whole structure vibrates at the same natural frequency. For convenience, suppose that the section constants of different members are the same, so the dimensionless frequencies $b_k, k = 1, \cdots, 4$ of different members are uniform, and in this paper it is denoted by b.

Only when $|\hat{A}| = 0$, the nontrivial solutions of Eq. (30) exist. Let $f(b) = |\hat{A}|$, then

$$f(b) = |\hat{A}| = 0. \tag{31}$$

Equation (31) is the characteristic equation of the impact system. It has unlimited positive real roots $b_n, n = 1,2,3,\cdots$, which correspond to the infinite natural frequencies of the system $p_n, n = 1,2,3,\cdots$,

$$p_n^2 = \frac{EI}{\rho AL^4} b_n^2. \tag{32}$$

For every $b_n, n = 1,2,3,\cdots$, the corresponding "characteristic vector" can be obtained:

$$\boldsymbol{d}_n = (D_{11,n}, D_{12,n}, D_{13,n}, D_{14,n}, D_{21,n}, D_{22,n}, D_{23,n}, D_{24,n}, D_{31,n}, D_{32,n}, D_{41,n}, D_{42,n})^{\mathrm{T}}. \tag{33}$$

In usual analysis of structure vibration, the characteristic vectors indicate the modal functions[5]. But, the "characteristic vectors" here are not the same as that just introduced. They are obtained by using the matrix theory to solve the matrix Eq. (30), which denotes the vectors composed of coefficients in every order of modal functions $V_{1,n}(\xi_1)$, $\Psi_{2,n}(\xi_1)$, $V_{2,n}(\xi_2)$, $\Psi_{2,n}(\xi_2)$, $U_{3,n}(\xi_3)$ and $U_{4,n}(\xi_4)$. Utilizing the relation between C_{ij} and D_{ij}, we can determine the modal functions corresponding to every characteristic root $b_n, n = 1,2,3\cdots$:

$$\begin{aligned} V_{1,n}(\xi_1) = {} & \frac{L\,\alpha_n}{b_n}\cdot\frac{1}{s^2-\alpha_n^2}\cdot D_{11,n}\cdot\cos(b_n\alpha_n\xi_1) - \frac{L\,\alpha_n}{b_n}\cdot\frac{1}{s^2-\alpha_n^2}\cdot D_{12,n}\cdot\sin(b_n\alpha_n\xi_1) \\ & + \frac{L\beta_n}{b_n}\cdot\frac{1}{s^2-\beta_n^2}\cdot D_{13,n}\cdot\cos(b_n\beta_n\xi_1) - \frac{L\beta_n}{b_n}\cdot\frac{1}{s^2-\beta_n^2}\cdot D_{14,n}\cdot\sin(b_n\beta_n\xi_1), \end{aligned} \tag{34}$$

$$\begin{aligned} \Psi_{1,n}(\xi_1) = {} & D_{11,n}\cdot\sin(b_n\,\alpha_n\xi_1) + D_{12,n}\cdot\cos(b_n\,\alpha_n\xi_1) \\ & + D_{13,n}\cdot\sin(b_n\beta_n\xi_1) + D_{14,n}\cdot\cos(b_n\beta_n\xi_1), \end{aligned} \tag{35}$$

$$\begin{aligned} V_{2,n}(\xi_2) = {} & \frac{L\,\alpha_n}{b_n}\cdot\frac{1}{s^2-\alpha_n^2}\cdot D_{21,n}\cdot\cos(b_n\alpha_n\xi_2) - \frac{L\,\alpha_n}{b_n}\cdot\frac{1}{s^2-\alpha_n^2}\cdot D_{22,n}\cdot\sin(b_n\alpha_n\xi_2) \\ & + \frac{L\beta_n}{b_n}\cdot\frac{1}{s^2-\beta_n^2}\cdot D_{23,n}\cdot\cos(b_n\beta_n\xi_2) - \frac{L\beta_n}{b_n}\cdot\frac{1}{s^2-\beta_n^2}\cdot D_{24,n}\cdot\sin(b_n\beta_n\xi_2), \end{aligned} \tag{36}$$

$$\begin{aligned} \Psi_{2,n}(\xi_2) = {} & D_{21,n}\cdot\sin(b_n\alpha_n\xi_2) + D_{22,n}\cdot\cos(b_n\alpha_n\xi_2) \\ & + D_{23,n}\cdot\sin(b_n\beta_n\xi_2) + D_{24,n}\cdot\cos(b_n\beta_n\xi_2), \end{aligned} \tag{37}$$

$$U_{3,n}(\xi_3) = D_{31,n}\cdot\cos(\Omega_n\xi_3) + D_{32,n}\cdot\sin(\Omega_n\xi_3), \tag{38}$$

$$U_{4,n}(\xi_4) = D_{41,n}\cdot\cos(\Omega_n\xi_4) + D_{42,n}\cdot\sin(\Omega_n\xi_4). \tag{39}$$

5 Dynamic Responses

With every order of characteristic root of the impact system $b_n, n = 1,2,3,\cdots$ corresponding to the characteristic vector $\boldsymbol{d}_n$, we can obtain the corresponding modal functions $V_{1,n}(\xi_1)$, $\Psi_{2,n}(\xi_1)$, $V_{2,n}(\xi_2)$, $\Psi_{2,n}(\xi_2)$, $U_{3,n}(\xi_3)$ and $U_{4,n}(\xi_4)$. For "normalizing" the modal functions, they are divided by a certain coefficient (for example $D_{11,n}L\,\alpha_n/(b_n(s^2-\alpha_n^2))$, the coefficient of $\cos(b_n\alpha_n\xi_1)$ in $V_{1,n}(\xi_1)$, which is denoted by A_n. Then we can get the "normalized" modal functions $\overline{V}_{1,n}(\xi_1)$, $\overline{\Psi}_{2,n}(\xi_1)$, $\overline{V}_{2,n}(\xi_2)$, $\overline{\Psi}_{2,n}(\xi_2)$, $\overline{U}_{3,n}(\xi_3)$ and $\overline{U}_{4,n}(\xi_4)$, in which all the terms are known except for A_n. And then every modal function can be expressed as the product of A_n and the corresponding "normalized" modal function.

Because of the symmetry of the structure and the loading, the beam has only a rigid translation displacement, without any rigid rotation displacement. For the whole structure, the rigid displacement can be expressed by $A_0 t + B_0$. So the transverse displacements and bending rotation angles of the impact system can be written in following series form:

1) For beam 1:

$$v_{1,n}(\xi_1,t) = A_0\cdot t + B_0 + \sum_{n=1}^{+\infty} A_n\cdot \overline{V}_{1,n}(\xi_1)\cdot \sin p_n t, \tag{40}$$

$$\psi_{1,n}(\xi_1,t) = \sum_{n=1}^{+\infty} A_n\cdot \overline{\Psi}_{1,n}(\xi_1)\cdot \sin p_n t. \tag{41}$$

2) For beam 2:

$$v_{2,n}(\xi_2,t) = A_0\cdot t + B_0 + \sum_{n=1}^{+\infty} A_n\cdot \overline{V}_{2,n}(\xi_2)\cdot \sin p_n t, \tag{42}$$

$$\psi_{2,n}(\xi_2,t) = \sum_{n=1}^{+\infty} A_n\cdot \overline{\Psi}_{2,n}(\xi_2)\cdot \sin p_n t. \tag{43}$$

3) For bar 3:

$$u_{3,n}(\xi_3,t) = A_0\cdot t + B_0 + \sum_{n=1}^{+\infty} A_n\cdot \overline{U}_{3,n}(\xi_3)\cdot \sin p_n t. \tag{44}$$

4) For bar 4:

$$u_{4,n}(\xi_4,t) = A_0\cdot t + B_0 + \sum_{n=1}^{+\infty} A_n\cdot \overline{U}_{4,n}(\xi_4)\cdot \sin p_n t. \tag{45}$$

6 Initial Conditions

Initial conditions are

$$v_1(\xi_1,t)\big|_{t=0} = 0;\ \psi_1(\xi_1,t)\big|_{t=0} = 0, \qquad 0 \leqslant \xi_1 \leqslant 1/2, \tag{46}$$

$$v_2(\xi_2,t)\big|_{t=0} = 0;\ \psi_2(\xi_2,t)\big|_{t=0} = 0, \qquad 0 \leqslant \xi_2 \leqslant 1/2, \tag{47}$$

$$u_3(\xi_3,t)\big|_{t=0} = 0, \qquad 0 \leqslant \xi_3 \leqslant 1/2, \tag{48}$$

$$u_4(\xi_4,t)\big|_{t=0} = 0, \qquad 0 \leqslant \xi_4 \leqslant 1/2, \tag{49}$$

$$g_1(\xi_1) = \left.\frac{\partial v_1(\xi_1,t)}{\partial t}\right|_{t=0} = \begin{cases} -S_0, & \xi_1 = 1/2, \\ 0, & 0 \leqslant \xi_1 < 1/2, \end{cases} \tag{50}$$

$$\left.\frac{\partial \psi_1(\xi_1,t)}{\partial t}\right|_{t=0} = 0,\ \left.\frac{\partial v_2(\xi_2,t)}{\partial t}\right|_{t=0} = 0,\ \left.\frac{\partial \psi_2(\xi_2,t)}{\partial t}\right|_{t=0} = 0,\ \left.\frac{\partial u_3(\xi_3,t)}{\partial t}\right|_{t=0} = 0, \tag{51}$$

$$\left.\frac{\partial u_4(\xi_4,t)}{\partial t}\right|_{t=0} = \begin{cases} -S_0, & \xi_4 = 1/2, \\ 0, & 0 \leqslant \xi_4 < 1/2. \end{cases} \tag{52}$$

From the displacement initial conditions (46), (47), (48) and (49), we have

$$B_0 = 0. \tag{53}$$

And the rotation angle initial conditions are satisfied automatically.

7 Coefficient of Rigid Response A_0

For the rigid modes and the elastics modes of the impact system, the orthogonality condition can be deduced by using Betti's law[5]:

$$\lim_{\varepsilon\to+0}\Big[\int_0^{1/2-\varepsilon} \overline{m}_1(\xi_1)\cdot\overline{V}_{1,0}(\xi_1)\cdot\overline{V}_{1,n}(\xi_1)\cdot d\xi_1 + \int_0^{1/2-\varepsilon} \overline{m}_4(\xi_4)\cdot\overline{U}_{4,0}(\xi_4)\cdot\overline{U}_{4,n}(\xi_4)\cdot d\xi_4\Big]$$

$$+ \int_0^{1/2} \overline{m}_2(\xi_2)\cdot\overline{V}_{2,0}(\xi_2)\cdot\overline{V}_{2,n}(\xi_2)\cdot d\xi_2 + \int_0^{1/2} \overline{m}_3(\xi_3)\cdot\overline{U}_{3,0}(\xi_3)\cdot\overline{U}_{3,n}(\xi_3)\cdot d\xi_3$$

$$+\overline{M}\cdot\overline{V}_{1,0}\left(\frac{1}{2}\right)\cdot\overline{V}_{1,n}\left(\frac{1}{2}\right)=0,\qquad n=1,2,3,\cdots,\tag{54}$$

where $\overline{m}_1(\xi_1)$, $\overline{m}_2(\xi_2)$, $\overline{m}_3(\xi_3)$ and $\overline{m}_4(\xi_4)$ are the distributed masses of elements 1, 2, 3 and 4, respectively; $\overline{M}$ is the concentrate rigid mass of the equivalent impact system.

Utilizing the initial conditions (46) – (52) and the orthogonality condition (54), the coefficient of rigid response A_0 can be obtained

$$A_0=-S_0/(3.5\cdot\lambda+1).\tag{55}$$

The coefficient A_0 is the rigid velocity of the structure $v_r=-S_0/(3.5\cdot\lambda+1)$. So the momentum of rigid response for the structure equals $(3.5\cdot\lambda+1)\cdot M_0\cdot v_r$, which is just equal to the initial momentum of the rigid-body $-M_0\cdot S_0$ before the contact-impact process. According to the principle of momentum conservation, the momentum sum of elastic responses for the system should be zero. From this view point, the rigid response of the structure can be evaluated directly.

8 Coefficients of Elastic Response A_n

For the elastic modes of different orders of the impact system, the similar orthogonality condition can be deduced by using Betti's law[5]. By using the same initial conditions and the orthogonality condition, the coefficients of elastic response A_n can be obtained

$$\begin{aligned}A_n=\;&\overline{M}\cdot\overline{V}_{1,m}\left(\frac{1}{2}\right)\cdot g_1\left(\frac{1}{2}\right)\Big/p_n^*\cdot\Big\{\lim_{\varepsilon\to+0}\Big\{\int_0^{1/2-\varepsilon}[\overline{m}_1(\xi_1)\cdot\overline{V}_{1,n}^2(\xi_1)+\overline{I}_1(\xi_1)\cdot\overline{\Psi}_{1,n}^2(\xi_1)]\cdot d\xi_1\\&+\int_0^{1/2-\varepsilon}\overline{m}_4(\xi_4)\cdot\overline{U}_{4,n}^2(\xi_4)\cdot d\xi_4\Big\}+\int_0^{1/2}[\overline{m}_2(\xi_2)\cdot\overline{V}_{2,n}^2(\xi_2)+\overline{I}_2(\xi_2)\cdot\overline{\Psi}_{2,n}^2(\xi_2)]\cdot d\xi_2\\&+\int_0^{1/2}\overline{m}_3(\xi_3)\cdot\overline{U}_{3,n}^2(\xi_3)\cdot d\xi_3+\overline{M}\cdot\overline{V}_{1,n}^2\left(\frac{1}{2}\right)\Big\}.\end{aligned}\tag{56}$$

9 Dynamic Responses

Substituting the generalized coefficients A_0 and A_n into Eqs. (40) – (45), we can get the translation displacement and bending rotation angle of each member for the structure. Then we can obtain further the other dynamic responses, such as the shear force $Q(\xi,t)$, the moment $M(\xi,t)$ and the axial force $N(\xi,t)$, *etc.*, which are as follows:

1) For beam 1:

$$\begin{aligned}Q_1(\xi_1,t)&=-kAG\cdot\left[\frac{1}{L}\cdot\frac{\partial v_1(\xi_1,t)}{\partial\xi_1}-\psi_1(\xi_1,t)\right]\\&=-kAG\cdot\sum_{n=1}^{+\infty}A_n\cdot\left[\frac{1}{L}\cdot\frac{\partial\overline{V}_{1,n}(\xi_1)}{\partial\xi_1}-\overline{\Psi}_{1,n}(\xi_1)\right]\cdot\sin p_nt,\end{aligned}\tag{57}$$

$$M_1(\xi_1,t)=EI\cdot\frac{1}{L}\cdot\frac{\partial\psi_1(\xi_1,t)}{\partial\xi_1}=EI\cdot\sum_{n=1}^{+\infty}A_n\cdot\frac{1}{L}\cdot\frac{\partial\overline{\Psi}_{1,n}(\xi_1)}{\partial\xi_1}\cdot\sin p_nt.\tag{58}$$

2) For beam 2:

$$\begin{aligned}Q_2(\xi_2,t)&=-kAG\cdot\left[\frac{1}{L}\cdot\frac{\partial v_2(\xi_2,t)}{\partial\xi_2}-\psi_2(\xi_2,t)\right]\\&=-kAG\cdot\sum_{n=1}^{+\infty}A_n\cdot\left[\frac{1}{L}\cdot\frac{\partial\overline{V}_{2,n}(\xi_2)}{\partial\xi_2}-\overline{\Psi}_{2,n}(\xi_2)\right]\cdot\sin p_nt,\end{aligned}\tag{59}$$

$$M_2(\xi_2,t) = EI\cdot\frac{1}{L}\cdot\frac{\partial\psi_2(\xi_2,t)}{\partial\xi_2} = EI\cdot\sum_{n=1}^{+\infty}A_n\cdot\frac{1}{L}\cdot\frac{\partial\overline{\Psi}_{2,n}(\xi_2)}{\partial\xi_2}\cdot\sin p_n t. \tag{60}$$

3) For bar 3:

$$N_3(\xi_3,t) = EA\cdot\frac{\partial u_3(\xi_3,t)}{\partial\xi_3} = EA\cdot\sum_{n=1}^{+\infty}A_n\cdot\frac{\partial\overline{U}_{3,n}(\xi_3)}{\partial\xi_3}\cdot\sin p_n t. \tag{61}$$

4) For bar 4:

$$N_4(\xi_4,t) = EA\cdot\frac{\partial u_4(\xi_4,t)}{\partial\xi_4} = EA\cdot\sum_{n=1}^{+\infty}A_n\cdot\frac{\partial\overline{U}_{4,n}(\xi_4)}{\partial\xi_4}\cdot\sin p_n t. \tag{62}$$

10 Conclusions

1) In this paper, the generalized Fourier-Series method is used to derive the analytical solutions of transient responses for an unrestrained planar frame structure subjected to the impact of a moving rigid-body. It is demonstrated that the modal method can also be used to successfully solve this type of impact problem.

2) Through derivation, it is found that the impact problems of unrestrained structures follow the same regularity as that in the first two chapters of Ref. [3]. The dynamic response of a impact structure are composed of two parts: rigid response and elastic response, and the rigid response of the structure depends on the mass ratio of the structure to the rigid-body; The momentum of the rigid response of the structure equals that of the moving rigid-body before impact. According to the principle of the momentum conservation, the momentum sum of the elastic response in the structure is always zero.

3) The method this paper can be used to derive the impact response formulas for the similar unrestrained frame structure.

References:

[1] Doyle J F. *Wave Propagation in Structures*[M]. Springer-Verlag, New York, 1989, 7 – 156.

[2] Pao Y H, Keh D C, Howard S M. Dynamic response and wave propagation in plane trusses and frames[J]. *AIAA J*, 1999, **37**(5): 594 – 603.

[3] Zheng Haitao. *Transient Impact Response and Stress Wave Propagation in Unrestrained Structures*[D]. Master Degree's Dissertation of Tongji University. Tongji University, Shanghai, 2002, 6 – 33 (in Chinese).

[4] Huang T C. The effect of rotary inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions[J]. *Journal of Applied Mechanics*, 1961, **28**(4): 579 – 584.

[5] Clough R W, Penzien J. *Dynamics of Structures*[M]. Wang Guangyuan *et al.* (transl). Science Press, Beijing, 1981, 204 – 216.